13-7 Inverse Trigonometric Functions

Main Ideas

- Solve equations by using inverse trigonometric functions.
- Find values of expressions involving trigonometric functions.

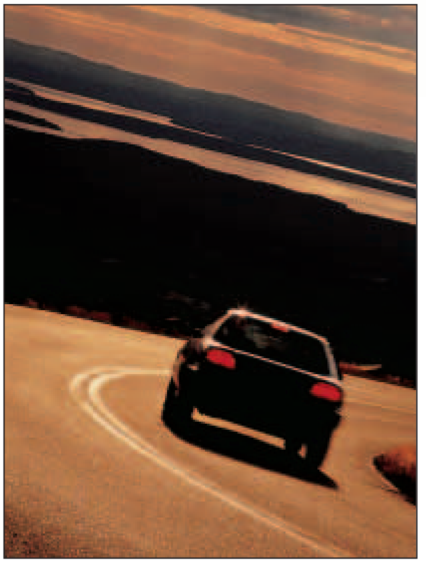
New Vocabulary

principal values
Arcsine function
Arccosine function
Arctangent function

GET READY for the Lesson

When a car travels a curve on a horizontal road, the friction between the tires and the road keeps the car on the road. Above a certain speed, however, the force of friction will not be great enough to hold the car in the curve. For this reason, civil engineers design banked curves.

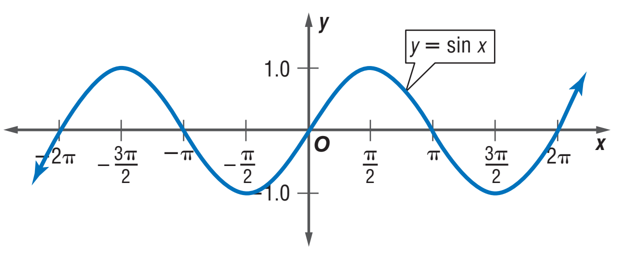
The proper banking angle θ for a car making a turn of radius r feet at a velocity v in feet per second is given by the equation $\tan \theta = \frac{v^2}{32r}$. In order to determine the appropriate value of θ for a specific curve, you need to know the radius of the curve, the maximum allowable velocity of cars making the curve, and how to determine the angle θ given the value of its tangent.

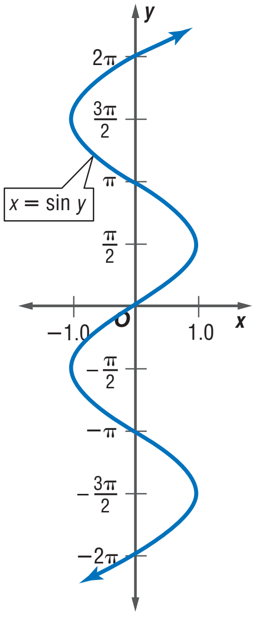


Solve Equations Using Inverses

Sometimes the value of a trigonometric function for an angle is known and it is necessary to find the measure of the angle. The concept of inverse functions can be applied to find the inverse of trigonometric functions.

In Lesson 8-8, you learned that the inverse of a function is the relation in which all the values of x and y are reversed. The graphs of $y = \sin x$ and its inverse, $x = \sin y$, are shown below.





Notice that the inverse is not a function, since it fails the vertical line test. None of the inverses of the trigonometric functions are functions.

We must restrict the domain of trigonometric functions so that their inverses are functions. The values in these restricted domains are called **principal values.** Capital letters are used to distinguish trigonometric functions with restricted domains from the usual trigonometric functions.

Doug Plummer/Photonica

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KEY CONCEPT — Principal Values of Sine, Cosine, and Tangent

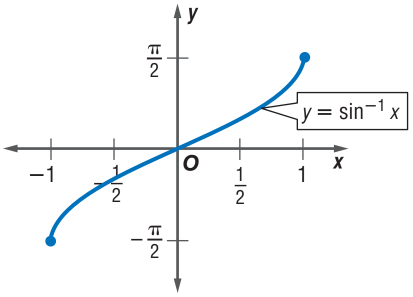
$y = \text{Sin } x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

$y = \text{Cos } x$ if and only if $y = \cos x$ and $0 \le x \le \pi$.

$y = \text{Tan } x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The inverse of the Sine function is called the **Arcsine function** and is symbolized **by Sin^{-1} or Arcsin**. The Arcsine function has the following characteristics.

- Its domain is the set of real numbers from -1 to 1.
- Its range is the set of angle measures from $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $\text{Sin } x = y$ if and only if $\text{Sin}^{-1} y = x$.
- $[\text{Sin}^{-1} \circ \text{Sin}](x) = [\text{Sin} \circ \text{Sin}^{-1}](x) = x$.



Study Tip

Look Back

To review **composition and functions**, see Lesson 7-1.

The definitions of the **Arccosine** and **Arctangent** functions are similar to the definition of the Arcsine function.

CONCEPT SUMMARY — Inverse Sine, Cosine, and Tangent

- Given $y = \text{Sin } x$, the inverse Sine function is defined by $y = \text{Sin}^{-1} x$ or $y = \text{Arcsin } x$.
- Given $y = \text{Cos } x$, the inverse Cosine function is defined by $y = \text{Cos}^{-1} x$ or $y = \text{Arccos } x$.
- Given $y = \text{Tan } x$, the inverse Tangent function is defined by $y = \text{Tan}^{-1} x$ or $y = \text{Arctan } x$.

The expressions in each row of the table below are equivalent. You can use these expressions to rewrite and solve trigonometric equations.

$y = \text{Sin } x$	$x = \text{Sin}^{-1} y$	$x = \text{Arcsin } y$
$y = \text{Cos } x$	$x = \text{Cos}^{-1} y$	$x = \text{Arccos } y$
$y = \text{Tan } x$	$x = \text{Tan}^{-1} y$	$x = \text{Arctan } y$

EXAMPLE 1 Solve an Equation

Solve $\text{Sin } x = \frac{\sqrt{3}}{2}$ by finding the value of x to the nearest degree.

If $\text{Sin } x = \frac{\sqrt{3}}{2}$, then x is the least value whose sine is $\frac{\sqrt{3}}{2}$. So, $x = \text{Arcsin } \frac{\sqrt{3}}{2}$.

Use a calculator to find x.

KEYSTROKES: 2nd [SIN^{-1}] 2nd [$\sqrt{\ }$] 3) ÷ 2) ENTER 60

Therefore, $x = 60°$.

CHECK Your Progress

1. Solve $\text{Cos } x = -\frac{\sqrt{2}}{2}$ by finding the value of x to the nearest degree.

Many application problems involve finding the inverse of a trigonometric function.

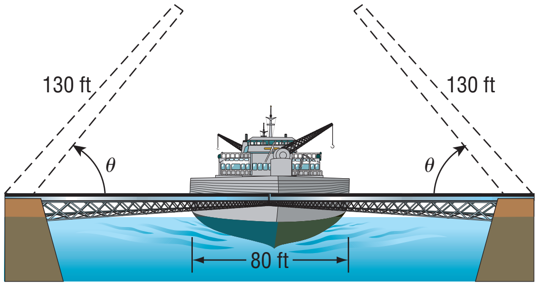
Real-World Link

Bascule bridges have spans (leaves) that pivot upward utilizing gears, motors, and counterweights.

Source: www.multnomah.lib.or.us

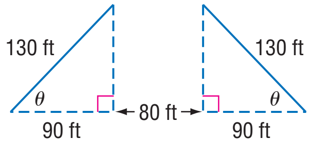
Real-World EXAMPLE — Apply an Inverse to Solve a Problem

2 **DRAWBRIDGE Each leaf of a certain double-leaf drawbridge is 130 feet long. If an 80-foot wide ship needs to pass through the bridge, what is the minimum angle θ, to the nearest degree, which each leaf of the bridge should open so that the ship will fit?**



When the two parts of the bridge are in their lowered position, the bridge spans 130 + 130 or 260 feet. In order for the ship to fit, the distance between the leaves must be at least 80 feet.

This leaves a horizontal distance of $\frac{260 - 80}{2}$ or 90 feet from the pivot point of each leaf to the ship as shown in the diagram at the right.



To find the measure of angle θ, use the cosine ratio for right triangles.

$\cos \theta = \frac{\text{adj}}{\text{hyp}}$ Cosine ratio

$\cos \theta = \frac{90}{130}$ Replace *adj* with 90 and *hyp* with 130.

$\theta = \cos^{-1}\left(\frac{90}{130}\right)$ Inverse cosine function

$\theta \approx 46.2°$ Use a calculator.

Thus, the minimum angle each leaf of the bridge should open is 47°.

CHECK Your Progress

2. If each leaf of another drawbridge is 150 feet long, what is the minimum angle θ, to the nearest degree, that each leaf should open to allow a 90-foot-wide ship to pass?

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Study Tip

Angle Measure

Remember that when evaluating an inverse trigonometric function the result is an angle measure.

Trigonometric Values You can use a calculator to find the values of trigonometric expressions.

EXAMPLE — Find a Trigonometric Value

3 **Find each value. Write angle measures in radians. Round to the nearest hundredth.**

a. ArcSin $\frac{\sqrt{3}}{2}$

KEYSTROKES: 2nd [SIN^{-1}] 2nd [$\sqrt{\ }$] 3) ÷ 2) ENTER 1.047197551

Therefore, ArcSin $\frac{\sqrt{3}}{2} \approx 1.05$ radians.

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b. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$

KEYSTROKES: [TAN] [2nd] [COS^{-1}] 6 [÷] 7 [ )] [ENTER] 0.6009252126

Therefore, $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right) \approx 0.60$.

CHECK Your Progress

3A. $\text{Arccos}\left(\frac{\sqrt{3}}{2}\right)$ **3B.** $\cos\left(\text{Arcsin}\frac{4}{5}\right)$

CHECK Your Understanding

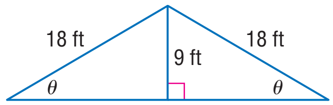
Example 1 (p. 807)

Solve each equation by finding the value of x to the nearest degree.

1. $x = \text{Cos}^{-1}\frac{\sqrt{2}}{2}$ **2.** $\text{Arctan } 0 = x$

Example 2 (p. 808)

3. ARCHITECTURE The support for a roof is shaped like two right triangles as shown at the right. Find θ.



Example 3 (pp. 808–809)

Find each value. Write degree measures in radians. Round to the nearest hundredth.

4. $\text{Tan}^{-1}\left(\frac{\sqrt{3}}{3}\right)$ **5.** $\text{Cos}^{-1}(-1)$ **6.** $\cos\left(\text{Cos}^{-1}\frac{2}{9}\right)$

7. $\sin\left(\text{Sin}^{-1}\frac{3}{4}\right)$ **8.** $\sin\left(\text{Cos}^{-1}\frac{3}{4}\right)$ **9.** $\tan\left(\text{Sin}^{-1}\frac{1}{2}\right)$

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–24	1
25–35	3
36, 37	2

Solve each equation by finding the value of x to the nearest degree.

10. $x = \text{Cos}^{-1}\frac{1}{2}$ **11.** $\text{Sin}^{-1}\frac{1}{2} = x$ **12.** $\text{Arctan } 1 = x$

13. $x = \text{Arctan}\frac{\sqrt{3}}{3}$ **14.** $x = \text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **15.** $x = \text{Cos}^{-1} 0$

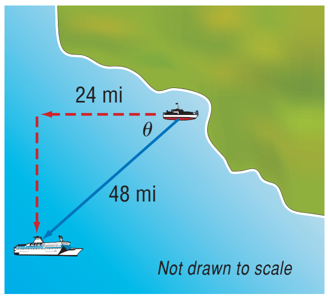
Find each value. Write angle measures in radians. Round to the nearest hundredth.

16. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$ **17.** $\text{Sin}^{-1}\frac{\pi}{2}$ **18.** $\text{Arctan}\frac{\sqrt{3}}{3}$

19. $\text{Arccos}\frac{\sqrt{3}}{2}$ **20.** $\sin\left(\text{Sin}^{-1}\frac{1}{2}\right)$ **21.** $\cot\left(\text{Sin}^{-1}\frac{5}{6}\right)$

22. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$ **23.** $\sin\left(\text{Arctan}\frac{\sqrt{3}}{3}\right)$ **24.** $\cos\left(\text{Arcsin}\frac{3}{5}\right)$

25. TRAVEL The cruise ship *Reno* sailed due west 24 miles before turning south. When the *Reno* became disabled and radioed for help, the rescue boat found that the fastest route to her covered a distance of 48 miles. The cosine of the angle at which the rescue boat should sail is 0.5. Find the angle θ, to the nearest tenth of a degree, at which the rescue boat should travel to aid the *Reno*.



EXTRA PRACTICE
See pages 922, 938.

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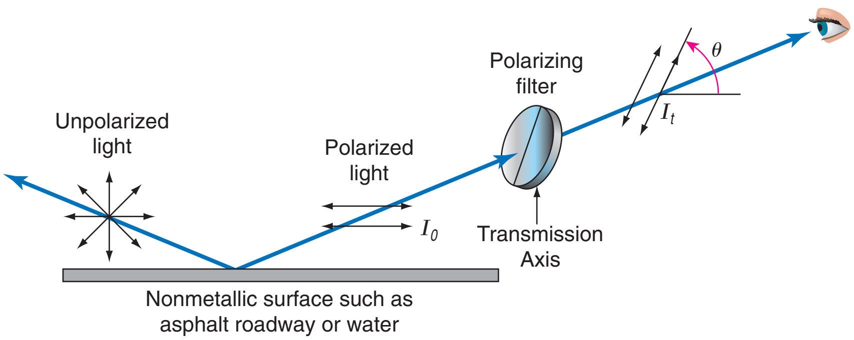
26. **OPTICS** You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose horizontally-polarized light with intensity I_0 strikes a polarizing filter with its axis at an angle of θ with the horizontal. The intensity of the transmitted light I_t and θ are related by the equation $\cos\theta = \sqrt{\frac{I_t}{I_0}}$. If one fourth of the polarized light is transmitted through the lens, what angle does the transmission axis of the filter make with the horizontal?



Find each value. Write angle measures in radians. Round to the nearest hundredth.

27. $\cot\left(\text{Sin}^{-1}\frac{7}{9}\right)$

28. $\cos\left(\text{Tan}^{-1}\sqrt{3}\right)$

29. $\tan(\text{Arctan } 3)$

30. $\cos\left[\text{Arccos}\left(-\frac{1}{2}\right)\right]$

31. $\text{Sin}^{-1}\left(\tan\frac{\pi}{4}\right)$

32. $\cos\left(\text{Cos}^{-1}\frac{\sqrt{2}}{2}-\frac{\pi}{2}\right)$

33. $\text{Cos}^{-1}(\text{Sin}^{-1} 90)$

34. $\sin\left(2\,\text{Cos}^{-1}\frac{3}{5}\right)$

35. $\sin\left(2\,\text{Sin}^{-1}\frac{1}{2}\right)$

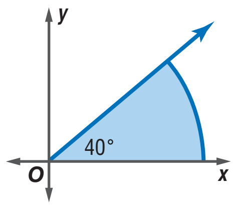
Real-World Link

The shot is a metal sphere that can be made out of solid iron. Shot putters stand inside a seven-foot circle and must "put" the shot from the shoulder with one hand.

Source: www.coolrunning.com.au

36. **FOUNTAINS** Architects who design fountains know that both the height and distance that a water jet will project is dependent on the angle θ at which the water is aimed. For a given angle θ, the ratio of the maximum height H of the parabolic arc to the horizontal distance D it travels is given by $\frac{H}{D}=\frac{1}{4}\tan\theta$. Find the value of θ, to the nearest degree, that will cause the arc to go twice as high as it travels horizontally.

37. **TRACK AND FIELD** A shot put must land in a 40° sector. The vertex of the sector is at the origin and one side lies along the x-axis. An athlete puts the shot at a point with coordinates (18, 17), did the shot land in the required region? Explain your reasoning.



For Exercises 38–40, consider $f(x) = \text{Sin}^{-1}x + \text{Cos}^{-1}x$.

38. Make a table of values, recording x and $f(x)$ for $x = \left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1, -\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}, -1\right\}$.

39. Make a conjecture about $f(x)$.

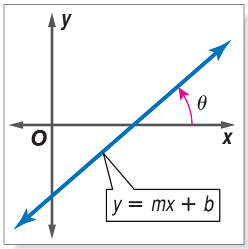
40. Considering only positive values of x, provide an explanation of why your conjecture might be true.

H.O.T. Problems

41. **OPEN ENDED** Write an equation giving the value of the Cosine function for an angle measure in its domain. Then, write your equation in the form of an inverse function.

Steven E. Sutton/PCN Photography

CHALLENGE For Exercises 42–44, use the following information.
If the graph of the line $y = mx + b$ intersects the x-axis such that an angle of θ is formed with the positive x-axis, then $\tan \theta = m$.



42. Find the acute angle that the graph of $3x + 5y = 7$ makes with the positive x-axis to the nearest degree.

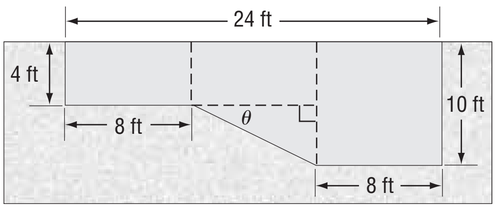
43. Determine the obtuse angle formed at the intersection of the graphs of $2x + 5y = 8$ and $6x - y = -8$. State the measure of the angle to the nearest degree.

44. Explain why this relationship, $\tan \theta = m$, holds true.

45. ***Writing in Math*** Use the information on page 806 to explain how inverse trigonometric functions are used in road design. Include a few sentences describing how to determine the banking angle for a road and a description of what would have to be done to a road if the speed limit were increased and the banking angle was not changed.

STANDARDIZED TEST PRACTICE

46. ACT/SAT To the nearest degree, what is the angle of depression θ between the shallow end and the deep end of the swimming pool?



Side View of Swimming Pool

A 25° **C** 53°

B 37° **D** 73°

47. REVIEW If sin θ = 23 and $-90° \le \theta \le 90°$, then $\cos(2\theta) =$

F $-\frac{1}{9}$.

G $-\frac{1}{3}$.

H $\frac{1}{3}$.

J $\frac{1}{9}$.

Spiral Review

Find the exact value of each function. (Lesson 13-6)

48. $\sin -660°$ **49.** $\cos 25\pi$ **50.** $(\sin 135°)^2 + (\cos -675°)^2$

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

51. $a = 3.1, b = 5.8, A = 30°$ **52.** $a = 9, b = 40, c = 41$

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function. (Lesson 6-7)

53. $f(x) = 5x^2 + 6x - 17$ **54.** $f(x) = -3x^2 + 2x - 1$ **55.** $f(x) = 4x^2 - 10x + 5$

56. PHYSICS A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. (Lesson 5-7)